Orbit theory, locally finite permutations and Morse arithmetic

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To the memory of my wife Rita

Abstract. The goal of this paper is to analyze two measure preserving transformation of combinatorial and number-theoretical origin from the point of view of ergodic orbit theory. We study the Morse transformation (in its adic realization in the group $\mathbf{Z}_2$ of integer dyadic numbers, as described by the author [*J. Sov. Math.* **28**, 667–674 (1985); *St. Petersburg Math. J.* **6** (1995), no. 3, 529–540]) and prove that it has the same orbit partition as the dyadic odometer. Then we give a precise description of time substitution of the odometer, which produces the Morse transformation. It is convenient to describe this time substitution in the form of random re-orderings of the group $\mathbb{Z}$, or in terms of random infinite permutations of the group $\mathbb{Z}$. We introduce the notion of *locally finite permutations (LFP) or locally finite bijection (LFB), and uniformly locally finite time substitution (ULFTS)* for the group $\mathbb{Z}$ (and for all amenable groups). Two automorphisms which have the same orbit partitions are called *allied* if the time substitution of one to another is ULFTS. Our main result is that the Morse transformation and the odometer are allied. The theory of random infinite permutations on the group $\mathbb{Z}$ (and on more general groups) is as strong as the ergodic theory of actions of the group. The main task in this area is the investigation of infinite permutations, and measures on the space of infinite permutations, as well as the study of linear orderings on $\mathbb{Z}$. The class of locally finite permutations is a useful class for such an analysis.

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1. Introduction

We consider in detail an example of time changing of a measure preserving transformation, namely the Morse transformation as a time-substitution of the odometer, which is the simplest ergodic transformation. The odometer is the operation of adding unity, $Tx = x + 1$, in the additive group $\mathbf{Z}_2$ of integer dyadic numbers. The main result is the calculation of a random re-ordering on the orbits of the odometer which presents the Morse transformation. The old observation of the author [**11, 6**] is that these two transformations have the same orbit partition in their natural realization in the space $\mathbf{Z}_2$; see also the recent paper [**9**]. Until now no explicit form of time substitution for a non-trival pair of automorphisms with the same orbit partitions was known. The odometer and the Morse transformation are perhaps the first useful example. Notice that the Morse transformation is a two-point extension of the odometer, and has non-discrete part in the spectrum, but a more delicate relation between them will become clear from the description of the corresponding time substitution. Namely, we prove that they are *allied* in the sense defined below.

In Section 2 we recall the main facts of orbit theory (Dye's theorem in particular), and formulate the general problem about possible reduction of ergodic theory to the theory of random permutations of the naturals numbers. Sections 3 and 4 are devoted to the Morse transformation and its realization as an adic transformation. This presentation allows us to prove that the orbit partitions of the odometer and of the Morse transformation are the same. The main result, which we explain in Section 5, consists of a description of the algorithm for the time substitution or re-ordering of the orbits of the odometer to the orbits of the Morse transformation. Finite substitutions of the set $\{2^n, 2^n+1, \dots, 2^{n+1}-1\}$ – so called Morse substitutions – play the key role in this algorithm. We emphasize the opening up of the possibility to study new models of measure preserving transformations, and new related tools. By this we mean that studying the group of permutations of the group $\mathbb{Z}$ and the space of linear orders on $\mathbb{Z}$, as well as invariant measures on those spaces. For example, we define a new relationship between two measure preserving transformations or between two actions of an amenable group with the same orbit partitions as follows. We say that the actions *are allied* if there is a time substitution of one action to another which is locally finite with the fixed set of finite permutations, which is the same for almost every orbit. In this case we called time substitution as *uniformly locally finite time substitution.* (ULFTS). We prove that *the odometer and the Morse transformation*

are allied. More precisely, the corresponding measure on the group of permutation of $\mathbb{Z}$ is concentrated on the permutations which almost preserve increasing sequences of intervals and finite permutations of intervals depends on the length of interval only. A more convenient language here is that of *re-ordering* of the group $\mathbb{Z}$. In general, we can say that *ergodic theory (for actions of the group $\mathbb{Z}$) can be considered as the theory of random permutation of the naturals, or as the theory of random linear orderings on the naturals.* The group acts on this space, and this action is universal in the sense that any action (up to isomorphism) can be realized in this way. The re-ordering of the orbit show us how different are the two transformations, or, how random is the re-ordering of their orbits. This point presumably is useful for automorphisms with positive entropy (in order to measure the difference between Bernoulli and non-Bernoulli automorphisms). The very special and ingenious structure (*of locally finite ordering*) in the case of the Morse transformation is also interesting as a new source of measures in the group of infinite permutations.

The similar definitions of locally finite bijection and allied actions can be done for an arbitrary amenable group.

2. Ergodic theory as analysis of infinite permutations

2.1. Dye's theorem and orbit theory. The simplest ergodic transformation is undoubtedly the 2-adic odometer (or "adding machine"):

$$T : x \mapsto x + 1, \quad x \in \mathbf{Z}_2,$$

where $\mathbf{Z}_2$ is the additive group of 2-adic integers equipped with Haar measure m. Indeed, T has discrete spectrum, comprising the group of all roots of unity of order $2^n, n = 0, 1 \ldots$. Although the structure of this automorphism is very simple, its orbit partition is universal in the class of ergodic measure-preserving transformations, as mentioned above.

Theorem 1 (H. Dye [**3**])**.** *For each ergodic measure preserving transformation M of the space (X, m), there exists a transformation S of the same space which is metrically isomorphic to the 2-odometer T and which has the same orbit partition as M, so that M is a* time change *of S:*

$$Mx = S^{t(x)}x,$$

where $t(\cdot)$ is a $\mathbb{Z}$-valued measurable function on X, and vice versa, there exists a measurable function $n(\cdot)$ such that $Sx = M^{n(x)}x$.

Isomorphism between T and S means the existence of an invertible measure-preserving map V such that $VTV^{-1} = S$, $V : X \to X$, and $V\mu = \mu$. So, by Dye's theorem, each ergodic automorphism is isomorphic to an automorphism of $\mathbf{Z}_2$, preserving Haar measure, and with the same orbit partition as the odometer.

The functions $t(\cdot), n(\cdot)$ are called *time-substitutions* or *jump-functions.*

So, the odometer has a universal orbit partition: the orbit partition of any ergodic transformation M is metrically isomorphic to the orbit partition of the odometer T. This partition is a standard hyperfinite countable homogeneous ergodic partition, that is the union of the decreasing ergodic sequence of the finite $2^n, n = 1, \dots$ homogeneous partitions[(1)].

By definition, the orbit partition of the odometer T is the partition into the cosets of the subgroup $\mathbb{Z}$ in the group $\mathbf{Z}_2$. We will prove that it coincides (mod 0) with the *tail partition* – that is, the partition into the cosets of the group $\sum_1^\infty(\mathbb{Z}/2)$ in the group $\prod_1^\infty(\mathbb{Z}/2)$. There is a difference between these two partition on the countable set $\mathbb{Z}$ due to the fact that positive and negative integers belong to different tail classes, namely the classes of $\{0\}^\infty$ and $\{1\}^\infty$ in the group $\mathbf{Z}_2$. Both the groups $\mathbf{Z}_2$ and $\prod_1^\infty(\mathbb{Z}/2)$ may be equipped with Haar measures, and become metrically isomorphic as measure spaces with those partitions. We will prove that the orbit partition of the Morse automorphism in its adic realization also coincides (mod 0) with the tail partition, and consequently there is a reversible time substitution which brings the Morse automorphism to the odometer. The goal of the paper is to study the arithmetic properties of that time substitution, and the so-called Morse arithmetic.

2.2. General theory of time substitutions. Suppose a countable group G acts as measure-preserving transformations on the measure space (X, μ). The orbit of the point $x \in X$ is the set $\{gx; g \in G\}$. The partition of the space (X, μ) into the orbits of the group G is called

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(1) We will not dwell on the interesting history of this theorem, which goes back to J. von Neumann and Murray [**7**] (where it is phrased in terms of the uniqueness of type-II_1 hyper-finite factors), and to several papers of H.Dye [**3**]. In [**10**] the author proved a lacunary theorem for homogeneous sequences of measurable partitions, and R. Belinskaya [**1**] used these as the main ingredient for a new proof of Dye's theorem (not known to us at that time), see also the later paper [**4**]. The final result in this direction was proved by Ornstein and Weiss [**8**] and Connes, Feldman, and Weiss [**2**]: all ergodic measure preserving actions of countable amenable groups have hyper-finite (or "tame") orbit partitions, isomorphic to the orbit partition defined by the odometer. Together with the previous results, this gives a characteristic property of measure-preserving actions of amenable groups.

the orbit (or trajectory) partition, and denoted by $\xi(G;X)$ or $\xi(G)$ if the action and space are fixed. In the case of the group $G=\mathbb{Z}$, we can write $\xi(T)$, where T is the generator of the action of the group $\mathbb{Z}$.

Assume that two measure preserving ergodic automorphisms T and M of the Lebesgue space (X,m) have the same orbit partitions,

$$\xi(T)=\xi(M)=\xi.$$

Consider the time-change function $t(\cdot)$ from above, defined by the formula:

$$Mx=T^{t(x)}x,$$

which exists because of the coincidence of orbit partitions. The function $t: X\to\mathbb{Z}$ is called a *time substitution* function from the automorphism T to the automorphism M or the "function of jumps from T to M".

Consider also the *complete time change* function $(x,k)\mapsto t(x,k)$, defined by

$$M^k x=T^{t(k,x)}x.$$

It is easy to express the function $t(k,x)$ using values of the function $t(x)$ on the same orbit as follows. We record the following formulas for $t(x,k)$:

$$\begin{aligned} t(0,x) &\equiv 0,\\ t(1,x) &= t(x),\\ t(k,x) &= \sum_{i=0}^{k-1} t(M^i x) \quad \text{for } k>0, \text{ and}\\ t(k,x) &= -\sum_{i=0}^{|k|-1} t(M^i x) \quad \text{for } k<0. \end{aligned}$$

For example,

$$M^{-1}x=T^{-t(M^{-1}x)}x$$

and

$$t(-1,x)=-t(M^{-1}x).$$

It is clear from the definition that M and T have the same orbit partition if the set of values of function $t(x,\cdot)$ when k run over $\mathbb{Z}$ coincides with $\mathbb{Z}$ for almost every x, or in other words $\{t(M^k x)\mid k\in\mathbb{Z}\}=\mathbb{Z}$ for a.a. x.

It is natural to look at the function $t(\cdot,\cdot)$ as a map from X to the group of all permutations of the integers – $\mathfrak{S}^{\mathbb{Z}}$. We denote this map by Θ_M, so

$$\Theta_M : x\mapsto\{k\mapsto t(x,k)\}\in\mathfrak{S}^{\mathbb{Z}}.$$

Evidently for almost all x the bijection $\Theta(x)$ of $\mathbb{Z}$ is an element of $\mathfrak{S}^{\mathbb{Z}}$.

We will call the permutation $\{k \to t(k,x)\} = \Theta(x) \in \mathfrak{S}^{\mathbb{Z}}$ the *time substitution from T to M at the point x*, or for brevity (if it is clear from the context what the maps T and M are) – the *substitution at the point x*. It is important not to confuse this permutation with the time substitution $t(x)$, which was defined above. The formulas above give the links between these two objects.

Define a subgroup $\mathfrak{S}_0$ of the group $\mathfrak{S}^{\mathbb{Z}}$ by

$$\mathfrak{S}_0^{\mathbb{Z}} \subset \mathfrak{S}^{\mathbb{Z}} : \mathfrak{S}_0^{\mathbb{Z}} = \{g \in \mathfrak{S}^{\mathbb{Z}} \mid g(0) = 0\}.$$

Since $t(0,x) \equiv 0$, the image of X under the map Θ_M lies in the subgroup $\mathfrak{S}_0^{\mathbb{Z}}$. Thus each point $x \in X$ is sent to an infinite permutation $\Theta(x) : k \mapsto t(x,k) \in \mathbb{Z}$, an element of the group $\mathfrak{S}^{\mathbb{Z}}$ of all infinite permutations of $\mathbb{Z}$.

Define an action of the group $\mathbb{Z}$ on $\mathfrak{S}_0^{\mathbb{Z}}$ by letting the generator D of $\mathbb{Z}$ act according to the formula

$$D(g)(k) = g(k+1) - g(1),$$

(D is the "modified shift"). The map D is a bijection of the subgroup $\mathfrak{S}_0^{\mathbb{Z}}$ onto itself (but is not a group isomorphism). The identity permutation Id and the reflection $-Id$ are both fixed points of the map D. We have defined an action of $\mathbb{Z}$, that is a dynamical system, on a subset of

$$\mathfrak{S}_0^{\mathbb{Z}} \subset \mathfrak{S}^{\mathbb{Z}} \mid \mathfrak{S}_0^{\mathbb{Z}} = \{g \in \mathfrak{S}^{\mathbb{Z}} \mid g(0) = 0\}.$$

We have defined a map from our system (X, m, M) to the group $\mathfrak{S}_0^{\mathbb{Z}}$ of all permutations of the integers with zero as a fixed point,

$$\Theta_M : X \to \mathfrak{S}_0^{\mathbb{Z}},$$

defined by

$$X \ni x \mapsto \Theta_M(x) \equiv \{k \mapsto t(k,x) \mid k \in \mathbb{Z}\}.$$

The image of the measure m on X under Θ_M is a measure $\Theta_M m = \mu_M$ on the group $\mathfrak{S}_0$. We define a new dynamical system using this as follows.

Theorem 2. *The map Θ_M is a homomorphism of the ergodic transformation M of the space (X, m) to the ergodic transformation D on the space $(\mathfrak{S}_0^{\mathbb{Z}}, \mu_M)$.*

If for almost all pairs (x,y) with $x, y \in X$ we have $t(\cdot, x) \neq t(\cdot, y)$, then Θ_M is an isomorphism onto the image of the triple (X, m, M) and

$$(\mathfrak{S}_0^{\mathbb{Z}}, \mu_M, D).$$

Thus we obtain (in the non-degenerate cases) a new model for the study of the automorphism M (relative to the automorphism T). The measure μ_M is interesting itself as a natural example of a measure on the group of all permutations of the integers (more generally, for a G-action, a measure on the group of permutations of G).

2.3. The space of linear orderings of $\mathbb{Z}$. In what follows it is more convenient to think of the time-substitution in terms of linear orders on $\mathbb{Z}$; more precisely *re-orderings* of the integers $\mathbb{Z}$. The image of the usual order on the group $\mathbb{Z}$ under the time substitution $\Theta_M(x)$ can be represented as follows:

$$\cdots \to t(-2,x) \to t(-1,x) \to t(0,x) = 0 \to t(1,x) = t(x) \to t(2,x) \to \cdots .$$

This is a *linear order of type $\mathbb{Z}$ on the group $\mathbb{Z}$*, depending on x. We will use the following notation from combinatorics: $a \lessdot b$ means that b is the immediate successor of a, or b immediately follows a with no intermediate elements. Using this, we can write

$$\cdots t(-2,x) \lessdot t(-1,x) \lessdot t(0,x) = 0 \lessdot t(x) = t(1,x) \lessdot t(2,x) \cdots .$$

Consider the space $\mathcal{T}$ of all linear orders on $\mathbb{Z}$ of type $\mathbb{Z}$, and equip that space with the natural weak topology and corresponding Borel structure. The group of shifts $\{S^n \mid n \in \mathbb{Z}\}$ acts on $\mathcal{T}$, and we can consider the shift-invariant Borel measures on the space $\mathcal{T}$. The *set of triples* $(\mathcal{T}, S, \mu)$ is once again a universal model in ergodic theory, as was the previous model of the group of all permutations. The formulas of the following lemma and its corollary show how to express this action.

Lemma 3. *For $k \geqslant 0$,*

$$t(k, Mx) = t(k+1, x) - t(x) \quad (= \sum_{i=1}^{k} t(M^i x)),$$

for $k < 0$, and

$$t(k, Mx) = t(k-1, x) - t(x) \quad (= -\sum_{i=1}^{|k|} t(M^{-i} x)).$$

Corollary 4. $t(M^k x) = t(k+1, x) - t(k, x)$.

Lemma 3 shows that the order

$$t(k,x) \lessdot t(k+1,x), k \in \mathbb{Z},$$

is sent to the order

$$t(k, Mx) \lessdot t(k+1, Mx) (= t(k+1,x) - t(x)), k \in \mathbb{Z},$$

or simply

$$t(k,x) - t(x) \lessdot t(k+1,x) - t(x)), k \in \mathbb{Z}.$$

This means that the new linear order induced by the automorphism M is simply the shift (translation) of the previous order. So the induced action of the group $\mathbb{Z}$ on the space $\mathcal{T}$ is defined independently of the automorphism M (recall that the function $t(\cdot)$ takes on all integer values). Thus the *action of* $\mathbb{Z}$ *on the space of linear orders* $\mathcal{T}$ *is simply the action by the shifts.* The usual order $k \lessdot k+1, k \in \mathbb{Z}$ and the opposite order $k \lessdot k-1, k \in \mathbb{Z}$ are fixed points of this action.

We are interested in the *shift-invariant measures on* $\mathcal{T}$. For a given automorphism T we can identify each automorphism M which has the same orbit partition as T with the shift invariant measure on the space $\mathcal{T}$. This measure is concentrated on the set of re-orderings of the orbits of T.

Thus all ergodic triples (X, M, m) up to isomorphism can be realized as the triple $(\mathcal{T}, S, \mu)$, where S is the shift on the space $\mathcal{T}$ and μ is a shift invariant measure on the $\mathcal{T}$. Remember that this isomorphism depends on the automorphism T, so this model as before can only give relative invariants of M with respect to T.

Problem 1. How does the class of invariant measures depend on the automorphism T?

2.4. The notion of locally finite permutation and locally finite ordering. Now we define a special class of permutations (or bijections) of the group $\mathbb{Z}$ (and, more generally, of countable amenable groups) and in parallel the corresponding special class of linear orders on the group $\mathbb{Z}$.

Definition 5. Locally finite bijections are defined as follows.

(1) A bijection $L : \mathbb{Z} \to \mathbb{Z}$ is called *locally finite* (LFB) if there exists an increasing sequence of intervals $I_1 \subset I_2 \subset \cdots$, with

$$\bigcup_{n \geqslant 1} I_n = \mathbb{Z},$$

with the property that for each $\epsilon > 0$ there exists $N = N(\epsilon)$ such that the intervals $I_n, n > N$ are L-invariant up to $\epsilon > 0$[(2)].

(2) More generally, let G be an arbitrary countable amenable group, and $L : G \to G$ be a bijection of G onto itself. Then L is called *locally finite* if there is an increasing exhaustive sequence of finite sets $\{I_n\}, \bigcup_n I_n = G$, where each I_n is a $\frac{1}{n}$-Følner set, (for some fixed choice of generators of G), with the property

[(2)]This means that $|I_n \Delta L(I_n)| < \epsilon |I_n|$.

that for each $\epsilon > 0$ there exists some $N = N(\epsilon)$ such that for all $n > N$ the sets I_n are L-invariant up to ϵ.

The property of being a locally finite bijection does not depend on the choice of generators for the group, and the class of locally finite bijections generates a subgroup of the group of all bijections on the group. To my knowledge nothing is known about this subgroup, and it would be interesting to investigate the algebraic structure of this subgroup.

It is easy to give examples of orderings which are not locally finite, but the locally finite class is most natural in analysis. We use this notion in the following paragraphs. The parallel definition for linear orderings on the group $\mathbb{Z}$ is the following.

Definition 6. A linear ordering $\prec$ on the group $\mathbb{Z}$ (of type $\mathbb{Z}$) is called *locally finite*(LFO) if there exists an increasing sequence of finite intervals $I_n = (a_n, b_n), \bigcup I_n = \mathbb{Z}$, with a linear order $\prec_n$ on each I_n such that the following holds.

(1) The linear order $(\mathbb{Z}, \prec)$ is a limit (stabilization in the natural sense) of $(I_n, \prec_n)$ when n tends to infinity.
(2) For each $\epsilon > 0$ there exists $N = N(\epsilon)$ with the property that for $n > N$ the restriction of $\prec_n$ to the interval $I_m, m < n$ coincides with $\prec_m$ for all elements of I_m except $\epsilon \cdot |I_m|$[(3)].

For the group $\mathbb{Z}$ the notion of locally finite ordering is consistent with the definition of locally finite bijection above. Namely, it is easy to check from the definition that if L is a locally finite bijection then the L-image of $>$ (that is, the order defined by $n \prec m \iff L^{-1}n > L^{-1}m$) is locally finite, and *vice versa.*

It is not difficult to prove (see [**13**]) the following fact:

Theorem 7. *Let S is ergodic automorphism which has the same orbit partition as odometer T. Then there exists time substitution of almost all orbits of T to the orbits of S which is locally finite for almost all orbits. The growth of the lengths of the corresponding intervals I_n depends on so called scale of automorphism S.*

Remark that by Dye's theorem and previous assertion each ergodic measure preserving transformation is isomorphic to another measure preserving transformation which has the same orbit as odometer and for which time substitution is locally finite for almost all orbits.

[(3)]In the sequel we will have a stronger condition, that the restriction of $\prec_n$ to I_m differs from $\prec_m$ at most on two points of I_m.

For Morse automorphism we will prove much more strong assertion about time substitution.

Definition 8. Suppose that for two measure preserving ergodic transformations T, M with the same orbits, and the locally finite time substitution from T to M is defined with permutations of each intervals I_n $n = 1, 2\dots$ (see definition) which are depended on n but are the same for almost all orbits. We call such time substitution as uniformly locally finite -ULFTS. (This is symmetric relation with respect to T, M. In this case we say that T, M are allied transformations.

We will call that Morse automorphism and odometer are allied, the time substitution has the exponential growth of length of interval and the finite permutation for each length of the interval I_n is so called the Morse permutation which is defined for each interval of length $2^n, n = 1, 2\dots$ and is the same for almost all orbits.

3. The odometer and the Morse automorphism

3.1. Orbit partition of the odometer. Assume that the space X is $\mathbf{Z}_2$, the integer dyadic numbers with Haar measure m and T is the transformation of adding unity in the group $\mathbf{Z}_2$,

$$T : x \to x + 1.$$

Notice that as a topological space (with pro-finite topology), and as a measure space (with Haar measure), the group $\mathbf{Z}_2$ is the same as the space of group $\prod_1^\infty \mathbb{Z}/2$ (with the $(\frac{1}{2}, \frac{1}{2})$ Bernoulli measure). Thus we can consider the natural action of the group $\sum_1^\infty \mathbb{Z}/2$ on $\mathbf{Z}_2$.

Let us consider the element $x \in \mathbf{Z}_2$ as a sequence $(x_1, x_2, \dots)$ with $x = 0, 1$, of the dyadic decomposition. Two sequences $\{x_k\}, \{y_k\}$ are called *cofinal* if $x_k = y_k$ for all $k > k_0(x, y)$. Finite sequences (that is, sequences which are cofinal with $(0)^\infty$ correspond to natural rational integers, sequences which are cofinal to $(1)^\infty$ correspond to negative rational integers. Each equivalence class of cofinal sequences is an orbit of the action of the group $\sum_1^\infty \mathbb{Z}/2$.

Lemma 9. (1) *The orbit partition of odometer T is the partition into the cosets of the subgroup of integer rational numbers $\mathbb{Z} \subset \mathbf{Z}_2$, and in terms of dyadic decompostion:*

(2) *The orbit partition $\xi(T)$ of the odometer T on the set $\mathbf{Z}_2 \setminus \mathbb{Z}$ coincides with the partition into cofinal classes (or with the orbit partition of the group $\sum_1^\infty \mathbb{Z}/2$. So these two partitions*

are the same $\pmod 0$*:*

$$\xi(T) = \xi(\sum_1^\infty \mathbb{Z}/2) \quad \textit{mod 0.}$$

The integers $\mathbb{Z}$ *generate one* T*-orbit, but decompose into two classes of cofinality – cofinal with* $(0)^\infty$ *and* $(1)^\infty$*.*

Proof. If $n \in \mathbb{Z}_+$ then evidently $T^n x = x + n$ and x are cofinal; if $x \neq 0$, then $x-1$ and x are cofinal. Consequently, if $x \in \mathbb{Z}$ and $n \in \mathbb{Z}$, then $x+z$ and x are cofinal. If x, y are cofinal then there exists $z \in \mathbb{Z}_+$ such that either $x + z = y$ or $y + z = x$. The orbit $\mathbb{Z}$ consists of two classes of cofinal sequences – one cofinal with 0^∞, and one with $(1)^\infty$. □

We can identify T-orbits of the generic point x of the group $\mathbf{Z}_2$ with the group $\sum_1^\infty \mathbb{Z}/2$, so it provides a linear order of type $\mathbb{Z}$ on the group $\sum_1^\infty \mathbb{Z}/2$.

Example 10. An interesting example of this nontrivial linear order is to use the point $x = -\frac{1}{3} \in \mathbf{Z}_2$, which has dyadic decompostion

$$-\tfrac{1}{3} = (10)^\infty$$

(see below). Its T-orbit is the set $\{\frac{1}{3} - k \mid k \in \mathbb{Z}\}$. On the other hand, the orbit is the class of all sequences cofinal with $(10)^\infty$, so it is possible to order by type $\mathbb{Z}$ this orbit, or *vice versa* to identify the group $\sum \mathbb{Z}/2$ which parameterizes that class in the natural sense, with the group $\mathbb{Z}$. The reader can do this easily.

Example 11. Notice that the group S_∞ also acts on the group

$$\mathbf{Z}_2 \simeq \prod_1^\infty \mathbb{Z}/2$$

as group permutations of the coordinates. However its orbit partition is finer than the partition into classes of cofinality, but does not coincide with it $\pmod 0$. The so-called Pascal automorphism [**11**] also acts on the group, and has the same orbit partition as the action of S_∞-infinite symmetric group.

If a measure-preserving transformation M of $\mathbf{Z}_2$ has the same orbit partition as the odometer T, then our formulas for M simplify. The function $X \ni x \mapsto t(x) \in \mathbb{Z}$ is

$$Mx = T^{t(x)}x = x + t(x),$$

(here addition is in the sense of the group $\mathbf{Z}_2$). We can simplify other formulas. For example,

$$M^k x = T^{t(k,x)} x = x + t(k,x) \pmod{\mathbf{Z}_2},$$

and so on.

At the same time, we believe that consideration of the Morse transformation below is a typical consideration in orbit theory. Namely, we will consider the well-known Morse system from the point of view of the dyadic odometer (that is, the relative invariants), and for this we will use the adic realization of the Morse automorphism, and the corresponding Morse permutations of the integers.

3.2. Traditional definition of the Morse automorphism. We recall the definition of the Morse automorphism. Consider the alphabet $\{0,1\}$ and define the *Morse substitution* by

$$\begin{aligned} \zeta(0) &= 01, \\ \zeta(1) &= 10. \end{aligned}$$

This is extended to all words in the alphabet $\{0,1\}$ by concatenation. The Thue–Morse sequence is a fixed point of the substitution,

$$u = u_0 u_1 u_2 \cdots = \lim_{n\to\infty} \zeta^n(0) = 0110100110010110\ldots.$$

The sequence u obeys the rule

$$u[0, 2^{n+1}-1] = u[0, 2^n - 1]\, \bar{u}[0, 2^n - 1] \quad \text{for } n \geqslant 0,$$

where $u[i,j] = u_i \cdots u_j$ and $\bar{u}[\ldots]$ denotes changing $0 \leftrightarrow 1$ inside $[\ldots]$.

The sequence u is non-3-periodic (no words appear 3 times repeatedly), and similarly is non-4-periodic and non-5-periodic. The sequence is well known in combinatorics, logic, and symbolic dynamics.

We next define the Morse automorphism. Consider the set of all two-sided infinite sequences in the alphabet $\{0,1\}$, all sub-words of which are the sub-words of the infinite Morse–Thue sequence u(4); this is the same set as the weak closure of the two-sided shifts of the Morse–Thue sequence u. This is a shift-invariant compact subset (in the weak topology) $\mathcal{M}$ of the space $Y = \{0,1\}^{\mathbb{Z}}$. The (left) shift on $\mathcal{M}$ is by definition the *Morse automorphism* of M as a topological space with action of $\mathbb{Z}$.

We recall the main properties of the Morse automorphism (see the earlier papers including [**5**], and more recent treatments [**9**]).

(4)In other words, this is the shift whose language coincides with the language defined by u.

Theorem 12. *The automorphism M is a minimal, uniquely ergodic, uniformly recurrent, transformation of $\mathcal{M}$. The spectrum of the corresponding unitary operator U_M on $L^2(\mathcal{M},\mu)$ (where μ is the unique invariant measure) is mixed, and contains a discrete part which is dyadic (and the same as for the 2-odometer) and a continuous part which is singular with respect to Lebesgue measure.*

4. The Morse transformation and its orbit partition

4.1. Adic realization of the Morse transformation. Consider the group $\mathbf{Z}_2$, the compact abelian group of the 2-adic integers under addition. It is clear that as a measure space this is isomorphic to the countable product of copies of the group $\mathbb{Z}/2$: $\mathbf{Z}_2 \cong \{0,1\}^{\mathbb{N}}$. The odometer transformation T is an *adic transformation* by definition, with respect to the natural partial (lexicographic) order on the group of 2-adic integers:

$$T : x \mapsto x+1.$$

We want to compare the Morse automorphism and the 2-odometer. For this we use the so called *adic realization* of the Morse automorphism. The general definition of an adic transformation requires an $\mathbb{N}$-graded graph (in our case, a graph with the two vertices 0 and 1 on all levels, and simple edges between any two vertices of adjacent levels), and with a linear order on the set of edges which end to a given vertex. For the odometer, the order is the same for both vertices: an edge which comes from 1 is greater than an edge which come form 0. In order to obtain the Morse transformation we *change the order of the symbols* $0,1$ *depending on the next symbol*,

$$0 \prec_0 1,\ \ 1 \prec_1 0$$

as follows:

$$\{x_i\} \prec \{y_i\} \quad \Longleftrightarrow \quad \exists\, j:\ x_i = y_i \text{ for } i > j$$

and

$$\{x_j\} \prec_z y_j,\ \text{ where } z = x_{j+1} = y_{j+1}.$$

This allows us to define a new partial order as the lexicographic order on the cofinal set of paths, or points of $\mathbf{Z}_2$ or set of all sequences of $0,1$.

We now show what the next point is in the sense of the new order. In order to define the next sequence to the sequence $x = (x_1, x_2, \dots) \in \mathbf{Z}_2$, $\quad x_i = 0,1$, we use the following procedure.

(1) Move through the sequence until the first repetition of symbols $x_i = x_{i+1}$, this will be either $\dots 00$ or $\dots 11$;
(2) Change $00 \to 10$, or $11 \to 01$;

(3) Substitute all the previous digits to ones, $11111\cdots 1$ in the first case, and correspondingly to zeros $00000\cdots 0$ in the second case:

$$\begin{array}{rcl} M((01)^n 00*) & = & (1^{2n+1}0*), \\ M((01)^n 1*) & = & (0^{2n}1*), \\ M((10)^n 0*) & = & (1^{2n}0*), \\ M((10)^n 11*) & = & (0^{2n+1}1*). \end{array}$$

For example,

$$\begin{array}{rcl} M(00*) & = & 10*, \\ M(0100*) & = & 1110*, \\ M(011*) & = & 001, \\ M(100*) & = & 110, \\ M(1011) = 0001 & & \end{array}$$

The rule expressing how to go from x to $M(x)$ may also be expressed as follows. Suppose that in a sequence

$$\mathbf{Z}_2 \ni x = (x_1, x_2, \dots x_r, x_{r+1}, \dots), x_i = 0, 1, i = 1 \dots$$

the first two adjacent coordinates which are equal are $x_r = x_{r+1}$.

This means that $x_1 = x_3 = x_5 = \dots = x_t \neq x_2 = x_4 = \dots x_s$, where either $t = r, s = r+1$ or $t = r-1, s = r$ depending on the parity of r.

If $x_r = 0$, then the sequence $M(x)$ has the first $r-1$ coordinates equal to 1, and all subsequent coordinates are not changed: $(1, 1 \dots 1, 0, *, *)$. In this case, in the formula $M(x) = x + t(x)$, we have $t(x) > 0$.

If $x_r = 1$ then $M(x)$ has the first $r-1$ coordinates equal to 0, and all subsequent coordinates are not changed: $(0, 0, \dots 0, 1, *, * \dots)$. In this case, in the formula $M(x) = x + t(x)$, we have $t(x) < 0$.

Our algorithm of definition of $M(x)$ does not work if x has finitely many 0s or 1s or equal adjacent coordinates with 0 and 1. Define as an *exceptional set of* $\mathbf{Z}_2$ the countable set of points of $\mathbf{Z}_2$ whose dyadic decomposition has finitely many adjacent pairs x_r, x_{r+1} of the type: $x_r = x_{r+1} = 0$ and $x_r = x_{r+1} = 1$, or (in a more direct description) the set of sequences which are cofinal to $(0)^\infty, (1)^\infty, (01)^\infty, (10)^\infty$.

Theorem 13. *The orbit partition of the Morse transformation M in its adic realization* $\pmod 0$ *coincides with the orbit partition of the odometer. More exactly, consider the M-orbit of the point $x \in \mathbf{Z}_2$ which has infinitely many coordinates with $x_r = x_{r+1}$, and infinitely many 0s and 1s (this occurs on a set of full measure). Then the orbit of x coincides with the set of all points which are cofinal with x. On the set*

of exceptional points (which has Haar measure 0*) the orbit equivalence relation of M is different from the relation of cofinality.*

Proof. We start the proof with a simple lemma.

Lemma 14. *Let $x \in \boldsymbol{Z}_2$ and assume that $x = \{x_1 \dots\}$ is the dyadic expansion of x. If for some r we have $x_r = x_{r+1}$, then each element y of $\boldsymbol{Z}_2$ for which $y_i = x_i, 1 < i < r$ belongs to the M-orbit of x.*

Proof. Suppose that $x_r = 0$. Then if we apply the definition of the Morse transformation to the sequence with coordinates

$$x_1 = x_2 = \cdots = x_n = 0,$$

we obtain all integers $0, 1 \dots 2^n - 1$, so the fragment of M-orbit of the point $x_k = 0, k = 1 \dots n$ coincides with the set $0, 1 \dots 2^r - 1$, consequently all y with the condition $y_i = x_i, 1 < i < r$ belongs to the M-orbit of the point $\mathbf{0}$. If $x_r = 1$ then the same is true if we start with a sequence with coordinates

$$x_1 = x_2 = \cdots = x_n = 1.$$

□

Returning to the proof of Theorem 13, we use the condition that there are infinitely many k with $x_k = 1$ in order to define correctly the full (two-sided) orbit of x (see the paragraph above about exceptional orbits). Now suppose that two points x and y are cofinal, and

$$x_k = y_k, k > N.$$

Then there exists $r > N$ such that $x_r = x_{r+1} = 0$ so, by Lemma 14, y belongs to the orbit of x, completing the proof. □

It is also possible to define the Morse transformation as adic transformation using *differentiation* of binary sequences. Define the differentiation operation

$$D : \mathbf{Z}_2 \to \mathbf{Z}_2$$

by

$$D(\{x_n\}_{n=0}^{\infty}) = \{(x_{n+1} - x_n) \bmod 2,\ n = 0, 1, \dots\}.$$

We remark that there are no good, simple "arithmetic" or "analytic" expressions for the behavior of D. The next result (see [**9**]) relates the Morse–Thue sequence to the operator D.

Lemma 15. *$T \circ D = D \circ M$.*

This is an immediate corollary of the definition of M. The new definition was made by the author as an example of the adic realization of the transformation (see [**6, 11**]). In the adic realization, the Morse transformation M is a 2-covering of the odometer in its algebraic form.

4.2. Jump function of the Morse automorphism, and Morse arithmetic. Now we give a precise expression for the "jump-function" $t(\cdot)$ in the formula $M(x) = x + t(x)$. Notice that the value of the function $t(x)$ depends on the finite fragment of x; more exactly, on the fragment $(x_1, \dots x_r), r = r(x)$ where $r = \min\{k \mid x_k = x_{k+1}\}$ (see above).

Define the sequence:

$$a_r = \begin{cases} \frac{2^{r+1}-1}{3} & \text{if } r \equiv 1 \pmod 2, \\ \frac{2^{r+1}-2}{3} & \text{if } r \equiv 0 \pmod 2) \end{cases}$$

for $r \geqslant 0$.

The first few values of a_r as a function of $r = 0, 1 \dots$ are shown below:

r	0	1	2	3	4	5	6	7	8	9	$\cdots$
a_r	0	1	2	5	10	21	42	85	170	341	$\cdots$

This sequence satisfies the recurrence relation

$$a_{r+1} = 2^r + a_{r-1}, \quad r = 0, 1, 2 \dots,$$

with the initial conditions $a_0 = 0, a_1 = 1$. Another evident relation is

$$a_{r-1} + a_r + 1 = 2^r.$$

We conclude that

$$2^{r-1} \leqslant a_r < 2^r,$$

so in particular there can only be one number a_r in the interval between two adjacent powers of 2.

The recurrence relation

$$a_{2n} = 2a_{2n-1} + 1; \quad a_{2n+1} = 2a_{2n}, \quad n = 1, 2 \dots; \quad a_0 = 0$$

is a corollary of the definition. Notice that the dyadic expansion a_r corresponds to the sequence

$$(01)^{\frac{r+1}{2}} \quad \text{for odd} \quad r > 1$$

and

$$(10)^{\frac{r}{2}} \quad \text{for even} \quad r > 0,$$

or equivalently

$$(01)^n = a_{2n+1}, \quad (10)^n = 1(01)^{n-1} = a_{2n}.$$

For each $x \in \mathbf{Z}_2$, we define $r = r(x)$ to be the minimal index of the coordinate for which the equality $x_r = x_{r+1}$ occurs for the first time (see above). Consequently, for all x which has a repetition 00 or 11 we have the following result.

Theorem 16. (1) *The transformation M is defined on the group $\boldsymbol{Z}_2$ of 2-adic integers. It is continuous at all but two points: $-\frac{1}{3}$ and $-\frac{2}{3}$. The transformation M preserves the Haar measure on $\boldsymbol{Z}_2$ and is metrically isomorphic to the Morse automorphism.*

(2) *The explicit formula for the Morse transformation on $x \in \boldsymbol{Z}_2$ is*

$$M(x) = \begin{cases} x + a_r, & \textit{if} \quad x_r = 0, \\ x - a_r, & \textit{if} \quad x_r = 1. \end{cases}$$

where $r = r(x)$.

Both cases can be expressed in the formula

$$M(x) = x + (-1)^{x_r} a_r, \quad \textit{where} \quad r = r(x).$$

If we compare this with the initial formula, we obtain

$$M(x) = T^{t(x)} x = x + t(x),$$

so

$$t(x) = M(x) - x = (-1)^{x_{r(x)}} a_{r(x)}.$$

Our formula can be applied to the integers as elements of the group of dyadic integers $\mathbb{Z} \subset \mathbf{Z}_2$ as follow. For $n \in \mathbb{Z}$, denote by $r(n)$ the minimal number of the digit in the dyadic decomposition of

$$n = \sum x_k 2^k, x_k = 0, 1$$

for which the value $x_{r(n)} = x_{r(n)+1} = \epsilon(n)$. For example,

$$\begin{aligned} r(n) &= 1, \quad \epsilon(n) = 0, \quad \text{if} \quad n \equiv 0 \pmod 4, \\ r(n) &= 1, \quad \epsilon(n) = 1; \quad \text{if} \quad n \equiv 3 \pmod 4, \\ r(n) &= 2, \quad \epsilon(n) = 0, \quad \text{if} \quad n \equiv 1 \pmod 8, \\ r(n) &= 2, \quad \epsilon(n) = 1; \quad \text{if} \quad n \equiv 6 \pmod 8. \end{aligned}$$

The general formula for $n, k = 1, 2 \ldots$ is the following:

$$r(n) = k, \quad \epsilon(n) = 0, \quad \text{if} \quad n \equiv a_{k-1} (\text{mod} \quad 2^{k+1})$$

and

$$r(n) = k, \quad \epsilon(n) = 1, \quad \text{if} \quad n \equiv -a_{k-1} - 1 (\text{mod} \quad 2^{k+1}).$$

The general formula for $M(x)$, which generalizes the formula above, is

$$M(n) = n + (-1)^{\epsilon(n)} a_{r(n)}.$$

Thus we obtain the *Morse order on the integers*: the following table illustrates this order on the semigroup $\mathbb{N}$ which is half the orbit of $\mathbf{0}$:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$M(n)$	1	3	7	2	5	15	4	6	9	11	31	10	13	8	12	14

The extension to negative integers is given by the relation

$$M(-n) = -M(n-1) - 1.$$

This give the order on the other half orbit corresponding to **1**:

-1	-2	-3	-4	-5	-6	-7
-2	-4	-8	-3	-6	-16	-5

These tables give the time substitution at the point **0**. We call this the *Morse order*.

The following sequences show us the Morse dynamics on the integers, namely what is the Morse re-ordering of the integers:

$$0 \to 1 \to 3 \to 2 \to 7 \to 6 \to 4 \to 5 \to 15 \cdots,$$

and for negative integers:

$$-1 \to -2 \to -4 \to -3 \to -8 \to -7 \to -5 \to -6 \cdots.$$

As we saw, the integers generate the exceptional (semi)-orbits M (see above and Section 4.1); our goal is to describe this re-ordering for generic orbits and to present the time substitutions in a more explicit form.

5. The structure of time substitutions; main construction

Now we are ready to give the answer to the question about general time substitutions for the Morse automorphism. The first step is the definition of very interesting finite permutations (elements of the groups S_{2^n}) which will be used to describe the re-ordering of the group $\mathbb{Z}$, and the time substitutions.

5.1. Definition of Morse permutations and Morse order. Our first construction concerns some special elements of the finite symmetric groups S_{2^n} which we call *Morse permutations*, and defines a linear order on the set $1, 2, \dots 2^n$.

For all $n > 1$ we will define by induction a permutation $g_n \in S_{2^n}$. It is convenient for this definition to consider the ordered set of integers: $\{2^n, 2^n+1, \dots, 2^{n+1}-1\}$, instead of $\{1, \dots 2^n\}$; later we will use it as permutations of an arbitrary linear ordered set on integers $b, b+1, b+2, \dots, b+2^n-1, b \in \mathbb{Z}$ (using the shift $i \to i+b, i = 2^n, \dots 2^{n+1}-1$).

Here are some of the first permutations (n=1,2,3) presented in cycle form:

$$\begin{aligned} g_1 &: (2,3), \\ g_2 &: (4,5,7,6), \\ g_3 &: (8,9,11,10,15,14,12,13). \end{aligned}$$

We define the Morse permutations for arbitrary n by induction. Suppose we have already defined g_{n-1}, as a permutation of the set

$$\{2^{n-1}, \dots, 2^n - 1\}.$$

Then g_n will be defined as a permutation of the set

$$\{2^n, \dots, 2^{n+1} - 1\}.$$

The action of g_n on the first half of the set $\{2^n, \dots, 2^n + 2^{n-1} - 1\}$ is defined as an "almost" shift of permutation g_{n-1} onto 2^n. More precisely,

$$g_n(i) = g_{n-1}(i - 2^{n-1}) + 2^{n-1}, \quad 2^n \leqslant i \leqslant 2^n + 2^{n-1} - 1$$

with one important exception:

$$g_n(a_{n+1}) = 2^{n+1} - 1$$

(remember that $2^n < a_{n+1} < 2^{n+1}$). On the second half $\{2^n + 2^{n-1}, \dots, 2^{n+1} - 1\}$ the action of g_n is also made of copies of the action of g_{n-1}, but slightly different:

$$g_n(i) = g_{n-1}(i - 2^n) + 2^n,$$

again with one exception:

$$g_n(2^n + a_n) = 2^n.$$

All these permutations are well-defined, and are cyclic permutations, nevertheless the first and second examined halves of the set $2^n \dots 2^{n+1} - 1$ are "almost invariant" in the sense that there is only one element of each half that has the image in the opposite half.

We will use the Morse permutations in order to define the linear order on the set. For this we need to *break the cycle at one point*. This point we will call the *first* (or the *minimal*) point, and its pre-image (in the cycle) as the *last* (or the *maximal*) point. We will write the cycle from the first point when it is already selected.

We now define two opposite linear orders on the set

$$\{2^n, 2^n + 1, \dots, 2^{n+1} - 1\},$$

breaking the cycle as follows. The order τ_n has minimal element $2^{n+1} - 1$ and the second order $\bar{\tau}_n$ has minimal element 2^n. It is clear from the definition that the maximal (or the last) element in the order τ_n is a_{n+1}, and in the order $\bar{\tau}_n$ the maximal element is $2^{n+1} + 2^n - a_{n+1} - 1$. The order $\bar{\tau}_n$ is simply the image of τ_n under reflection $i \leftrightarrow 2^{n+1} + 2^n - i - 1$. Recall that the symbol $a \lessdot b$ means that b is next to a in the sense of the order.

The structure of the Morse permutation and order will be more transparent if we divide the set $\{2^n, \dots, 2^{n+1} - 1\}$ into groups of four

elements. We will see that there are two types (τ and $\bar{\tau}$) of such groups which are alternate.

Example 17.

$$\tau_3: \quad 15 \lessdot 14 \lessdot 12 \lessdot 13 \lessdot 8 \lessdot 9 \lessdot 11 \lessdot 10,$$

$$\bar{\tau}_3: \quad 8 \lessdot 9 \lessdot 11 \lessdot 10 \lessdot 15 \lessdot 14 \lessdot 12 \lessdot 13.$$

5.2. Random linear order on the group $\mathbb{Z}$, and time substitution for the Morse transformation. We want to define an explicit linear order on the group $\mathbb{Z}$ depending on x which corresponds to the time substitution from the odometer to the Morse transformation:

$$\cdots \to t(-2,x) \to t(-1,x) \to t(0,x) = 0 \to t(1,x) = t(x) \to t(2,x) \to \cdots,$$

where we recall that $M^k x = T^{t(k,x)}$. The values of $t(k,x)$ for a fixed generic x run over all of the group $\mathbb{Z}$. Thus we want to reorder the orbit of T to the orbit of M.

Definition 18. The Morse random order $\tau(x)$ (corresponding to the non-exceptional point x) on the group $\mathbb{Z}$ is the linear order defined using the map $k \mapsto t(k,x)$ where $M^k x = T^{t(k,x)} x$. In other words, it is the re-ordering of the T-order to the M-order on the orbit of the point x.

We will give an implicit description of the Morse order (depending on x). Firstly, we describe the structure of our answer, and explain what the Morse linear order $\tau(x)$ looks like.

Definition 19. A *Morse linear order* τ on the group $\mathbb{Z}$ comprises the linear orders on the systems of countably many finite intervals in $\mathbb{Z}$, each of length a power of two, the union of which is the whole group. On each of the finite intervals, the linear order follows the Morse order defined above, and we glue the boundary elements (the maximal and minimal points of the adjacent intervals).

Such an ordering, and such a corresponding infinite permutation, belongs to the class of *locally finite linear orderings of* $\mathbb{Z}$ defined in Section 2.4. The order depends on the point x and is therefore called a *random order.*

The locally finite ordering has a system of increasing intervals on $\mathbb{Z}$ of length 2^k for various k, and we equip each of these with its Morse order. At each stage it appears that the old interval is included in the new interval. The final points of each linear order τ_n are glued to one of the boundary points of the next interval. The length and order

of gluing hardly depends on the non-exceptional point x, indeed the structure of the construction is universal.

We next describe in more detail such a construction.

5.3. A parametrization of the points in $\mathbf{Z}_2$. It is convenient to parameterize the points $x \in \mathbf{Z}_2$ as follow. Let $x = (x_1, x_2 \dots)$ be the dyadic decomposition of x. The sequence of coordinates is a sequence of independent $0, 1$ variables with probability $(\frac{1}{2}, \frac{1}{2})$. Instead of the coordinates of x, we consider all the numbers $r_1(x), r_2(x) \cdots$ of coordinates for which $x_{r_i} = x_{r_i+1}$, and fix also the value 0 or 1 of $x_{r_1} = \epsilon(x)$. For almost all elements x the sequence $\{r_n\}$ is an infinite increasing sequence, which together with $\epsilon(x) = 0, 1$ defines x uniquely. Indeed, it is easy to see that if we know $\{r_n\}$ and $\epsilon(x)$ then we can restore all the subsequent coordinates[5]. The probabilistic properties of the parameters can be obtained from the fact that the Haar measure is the Bernoulli measure.

Lemma 20. (1) *The differences $r_{n+1} - r_n$ are mutually independent and have the same geometrical distribution*

$$Prob\{r_n = k\} = 2^{-k}, k = 1, \dots .$$

(2) *Consider the function $t(x)$ from Section 2.2 with $Mx = T^{t(x)}x$. Then*

$$Prob\{t(x) = \pm a_r\} \equiv Prob\{\Theta(x)(\mathit{1}) = \pm a_r\} = Prob\{r_1(x) = r\} = \tfrac{1}{2^r}.$$

This means that the values of $t(x)$ are not arbitrary, and have exponentially decreasing probability.

Fix x; for each such choice, corresponding data $\{r_n(x) = r_n\}_{n \in \mathbb{N}}$, and $\epsilon(x) = \epsilon$, we make a corresponding re-ordering of the group $\mathbb{Z}$. The point x will correspond to 0 in the group $\mathbb{Z}$, and the ordinary order on the group $\mathbb{Z}$ corresponds to the dynamics of the odometer $k \backsim T^k x$.

5.4. Construction of the Morse linear order, and its time-substitution. Now we describe the algorithm which sequentially constructs, for each fragment of dyadic numbers, the final interval on $\mathbb{Z}$ with the needed linear order.

(1) The initial interval is constructed as follows. Consider $r_1(x)$, which is greater than or equal to 1 by definition. If $r_1 = 1$, then $x = (0, 0*, * \dots)$ or $x = (1, 1, *, * \dots)$. In the first case, $Mx = Tx = (1, 0, *, * \dots)$, so the M-image of x (of 0)

[5]The sequence $\{r_n(x)\}$ can be interpreted easily in terms of the differentiation operation D from Section 3.2, since $r_n(x)$ is the place on which the sequence Dx has a 1.

is the same as the T-image and is 1, thus 1 is M-next to 0. In the second case $Mx = T^{-1}x$, so that -1 is M-next to 0. Thus the initial interval is either $\{0,1\}$ with the usual linear order, or $\{-1,0\}$ with that linear order. Suppose now that $r_1 = r > 1$. In this case the initial $(r-1)$-fragment of x has coordinates $(01)^s, (10)^s, 1(01)^s$ or $0(10)^s$, where $s = \frac{r-1}{2}$ or $s = \frac{r-2}{2}$ depending on the parity of r. We will consider 2^r points from $\mathbf{Z}_2$ whose coordinates with indices $m > r$ are the same as the coordinates of x. The set of these points in $\mathbb{Z}$ is the interval $0, 1, \dots 2^r$, and we will shift this set in order that the fragment $x_1, \dots x_r$ of the point x starts on the place of 0 in $\mathbb{Z}$. We translate the interval of integers $\{0, 1 \dots 2^r - 1\} \subset \mathbb{Z}$ to the interval of integers

$$\bar{I}_1 = \{-a_{r-1}, -a_{r-1}+1, \dots 0, \dots, 2^r - a_{r-1} - 1 = a_r\}$$

if $x_r = 0$, or

$$\bar{I}_1 = \{-a_r, -a_r + 1, \dots, 0, \dots, 2^r - a_r - 1 = a_{r-1}\}$$

if $x_r = 1$. These are the initial intervals of our construction, namely

$$I_1 = (-a_{r-1}, a_r)$$

and

$$\bar{I}_1 = (-a_r, a_{r-1}).$$

In the construction we have only used the fragment

$$(x_1, x_2, \dots x_{r+1})$$

of the point x. We must define a new linear order on this set according to the action of the Morse transformation.

(2) Now we apply the Morse order τ_r on the interval I_1 and $\bar{\tau}_r$ on the interval $\bar{I}_1$ defined above, using the shift of $2^r, 2^{r+1} - 1$ to those two intervals, as mentioned in the previous item. We obtain a linear order on the intervals depending on whether x_r is 0 or 1. Carrying the boundary points from the interval

$$2^n, \dots, 2^{n+1} - 1$$

to the interval which we obtained gives the following boundary points of our linear order. Initial (minimal) points are

$$-a_{r-1} \in (I_1, \tau(r)),$$

correspondingly

$$a_{r-1} \in (\bar{I}_1), \bar{\tau}(r)),$$

and the maximal (or last) points are

$$a_{r-1}+1 \in (I_1, \tau(r)),$$

correspondingly

$$-a_{r-1}-1 \in (\bar{I}_1, \bar{\tau}(r)).$$

We have obtained the initial step of the construction of the linear order $\tau(x)$.

Example 21. Let $r_1 = 3$. Then $x = (0,1,0,0,*,\dots)$ or $x = (1,0,1,1,*\dots)$, and one of the ends of the interval will be $a_r = 5$, and we obtain the interval

$$(-2,-1,0,1,2,3,4,5)$$

for the first case and

$$(-5,-4,-3,-2,-1,0,1,2)$$

for the second case. The initial points are -2 and $+2$ and the last points are 3 and -3 correspondingly. The linear orders are τ and $\bar{\tau}$, namely

$$-2 \to -1 \to 1 \to 0 \to 5 \to 4 \to 2 \to 3,$$

or

$$2 \to 1 \to -1 \to 0 \to -5 \to -4 \to -2 \to -3.$$

We have given an explicit form of the first part of the time substitution for each point, so we already have an algorithm for calculating the function $t(\cdot)$. Because this function uniquely defined the functions $t(k,\cdot)$ by the formula

$$t(k,x) = \sum_{i=1}^{k-1} t(M^i x),$$

it is possible to stop the algorithm for constructing the time substitution here. However we will give a continuation in order to describe it as a random re-ordering of the whole group $\mathbb{Z}$.

(3) The general inductive step may be described as follows. Suppose that we have already considered the first $n-1$ members of the sequence $r_{n-1}(x) \equiv r_{n-1}$, and obtained the linear order of one of the intervals $I_{n-1}(x) = \{b_{n-1}, c_{n-1}\}$ of length $2^{r_{n-1}}$ and including $0 \in \mathbb{Z}$ which is equipped with the Morse linear order $\tau_{r_{n-1}}$ or $\bar{\tau}_{r_{n-1}}$ with the minimal point of the order coincided with one of the endpoints of I_n, either b_{n-1} or c_{n-1} correspondingly. We will choose the next interval $I_n(x) = \{b_n, c_n\}$ and define a linear order on it so as to include $I_{n-1}(x)$, and such that the restriction of the linear order on $I_{n-1}(x)$ coincides

with the initial linear order. Consider the number r_n, being the next after r_{n-1} with equal coordinates $x_{r_n} = x_{r_n+1}$. Denote the maximal point of the order on I_{n-1} by l_{n-1}. There are two cases:

(a) If $r_n = r_{n-1} + 1$, then $x_{r_n} = x_{r_{n+1}}$ and in this case $I_n = I_{n-1} \cup J$, the minimal element of I_n is the same as in I_{n-1}, where J is an interval adjoining I_{n-1} from the side which is opposite to the minimal element. The linear order on I_n has the same type τ or $\bar{\tau}$ as on I_{n-1}. The next element to the maximal element of I_{n-1} in I_n will be the second (non-minimal) endpoint of I_n.

(b) If $r_n > r_{n-1} + 1$ then the construction depends on the parity of the difference $r_n - r_{n-1} > 1$. If this difference is odd, then $x_{r_{n-1}} \neq x_{r_n}$ and the Morse order on I_n will change its type to be opposite to the type of order I_{n-1}; in particular the minimal element of I_n, say b_n, will be on the opposite side to the minimal element c_n of I_{n-1}. If the difference is even, then minimal elements are both b or both c. Now the interval $I_n = J \cup I_{n-1} \cup J'$ where J, J' are adjacent to the I_{n-1} intervals in $\mathbb{Z}$. The lengths of J and of J' are equal to

$$|J| = \sum_{i: r_{n-1}+1 < i < r_n - 1; i \equiv r_{n-1} (2)} 2^i;$$

and

$$|J'| = \sum_{i: r_{n-1}+1 < i < r_n - 1; i \equiv 1 + r_{n-1} (2)} 2^i.$$

It is clear that the length of I_n is 2^{r_n}. By definition, the restriction of the Morse order on I_n onto the interval I_{n-1} coincides with the initial order on I_{n-1}. The next element to the maximal element of I_{n-1} in I_n will be the second (nonminimal) endpoint of the interval I_n.

The randomness of the construction and of the re-ordering consists in the various possibilities for the sequence $r_n(x)$ and the values x_{r_n}. Thus the re-ordering can be different for various values of x. Nevertheless the structure of the new orders are similar for all points. Now it is evident that the probabilistic behavior of the length of the intervals I_n depends precisely on the sequence r_n and the size of jumps has geometrical distribution (see Lemma 20). For example, the long jumps have exponentially small probability.

5.4.1. *An exercise and an informal explanation.* A good concrete example of the ordering of $\mathbb{Z}$ is given by rational (periodic) elements of $\mathbf{Z}_2$. We will give the first fragment of the linear order for[(6)]

$$x = (100)^\infty = -\tfrac{1}{7}.$$

and leave for the reader the case of

$$x = (1100)^\infty = -\tfrac{1}{5}.$$

The orbit of the point $-\frac{1}{7} = (100)^\infty$ is interesting because it is a generic point (in the sense that the orbit is of type $\mathbb{Z}$), and at the same time the values of the coordinates which are repeated are the same, so $x_{r_k} = 0$. In the case of $(1100)^\infty$ the situation is more complicated: the values of x_{r_k} change as $x_{r_{2k+1}} = 1, x_{r_{2k}} = 0$. The periodicity does not give any simplifications, so together both cases give the full picture.

This is the beginning: the invariant 32 digits of the linear order for $-\frac{1}{7} = (100)^\infty$ are

$$\{\ldots|\lessdot -9 \lessdot -8 \lessdot -6 \lessdot -7, |\lessdot -2 \lessdot -3 \lessdot -5 \lessdot -4|, \lessdot +6 \lessdot +5 \lessdot +3 \lessdot +4|$$

$$\lessdot -1, \lessdot 0 \lessdot 2, \lessdot 1|, \lessdot 22 \lessdot 21 \lessdot 19 \lessdot 20|, \lessdot 15 \lessdot 16 \lessdot 18 \lessdot 17|, \lessdot$$

$$\lessdot 7 \lessdot 8 \lessdot 10 \lessdot 9| \lessdot 14 \lessdot 13 \lessdot 11 \lessdot 12| \ldots\}.$$

Here $a \lessdot b$ means that $Mx = T^a x, M^2 x = T^b x$, for example $MT^4 = T^6$ and $MT^6 = T^5$.

We can see that in all the examples the order subdivided (as marked by $|$) into the blocks with 4 points with order $(1, 2, 4, 3)$ or $(4, 3, 1, 2)$, then the 4 blocks generate the block of the next level, and so on. But the distance between the quadruples (or jumps) depends on x, more exactly on the number k of adjacent coordinates which have the same values $r_k = r_{k+1}$.

The cases $x = (0)^\infty, (1)^\infty, (01)^\infty, (10)^\infty$ remain. As we will see in the next section, these are one-sided: the first two left-sided, and the second two right-sided.

[(6)] This equality is true because

$$\underbrace{(100)^\infty + \cdots + (100)^\infty}_{7 \text{ terms}} = (1)^\infty = -1.$$

Similar arguments give the equalities $(01)^\infty = -\frac{2}{3}, (10)^\infty = -\frac{1}{3}, (1100)^\infty = -\frac{1}{5}$ and so on.

5.5. Addendum: Exceptional orbits. The points

$$\mathbf{0} = (0)^\infty, \mathbf{\text{-}1} = (1)^\infty$$

are exceptional: they have no full orbits because they have no pre-images. So the semi-orbit of $\mathbf{0}$ defines a linear order (and a permutation) on the semigroup $\mathbb{Z}_+$, and the semi-orbit of $\mathbf{1}$ similarly defines a linear order on $\mathbb{Z}_-$ (see the formulas at the end of Section 4.1). The points with denominator 3 are also exceptional and not generic, as we saw. The orbit of the point $(10)^\infty$ does not coincide with an orbit of the odometer. Now we can obtain the complete comparison of the orbit partition on $\mathbf{Z}_2$ for the odometer T and the Morse transformation M. We remark that for the odometer the points $\mathbf{0} = (0)^\infty$ and $\mathbf{1} = (1)^\infty$ belong to one orbit, but for the Morse transformation this is not true.

Consider the following orbits of the odometer T:

$$\begin{array}{ll} \tau_0, & \text{the orbit of } \mathbf{0}; \\ \tau_1, & \text{the orbit of } \frac{3k+1}{3}, k \in \mathbb{Z}; \\ \tau_2, & \text{the orbit of } \frac{3k+2}{3}, k \in \mathbb{Z}. \end{array}$$

On the other side, the Morse transformation M has two positive semi-orbits (which have no past), namely the orbits of the points 0^∞ and 1^∞, and two negative semi-orbits (which have no future), namely the orbits of the points $(01)^\infty$ and $(10)^\infty$. We call them *maximal points*, and denote the set of these two points as $MAX = \{(10)^\infty, (01)^\infty\}$. We will join those three orbits of the odometer T with four semi-orbits of M and obtain two new orbits of the Morse automorphism M by definition. By the initial definition, the orbit τ_0 divides into two positive semi-orbits of M – one starts with 0 and the second with -1: each of these glue correspondingly with orbits τ_1 and τ_2 of T, and recall that for the initial definition of M they are only negative semi-orbits. So we glue each two positive semi-orbits to two negative semi-orbits, and obtain two new full orbits of M.

Because the transformation M is not defined on these two sequences, we can by definition choose the values among another two sequences which conversely have no *pre-images*, or for which the inverse map M^{-1} is not defined. There are only two such points z for which there is no x with the property $M(x) = z$, namely $(0)^\infty = 0 \in \mathbb{Z}$ and $(1)^\infty = -1 \in \mathbb{Z}$.

We may assume[(7)] that

$$M((10)^\infty) \equiv M(-\tfrac{1}{3}) = (0)^\infty = \mathbf{0}, \quad M((01)^\infty \equiv M(-\tfrac{2}{3}) = (1)^\infty = -\mathbf{1}.$$

[(7)]We glued the semi-orbit of $\mathbf{0}$ to the semi-orbit of $-\frac{1}{3}$, and the semi-orbit of $\mathbf{1}$ to $-\frac{2}{3}$, but we can change this gluing to the opposite one.

Here the boldface numbers denote rational integers: $\mathbf{0}, \mathbf{1} \in \mathbb{Z} \subset \mathbf{Z}_2$.

This gives us the following picture:

$$\cdots + \tfrac{7}{3}, +\tfrac{10}{3}, +\tfrac{16}{3}, +\tfrac{13}{3}, -\tfrac{17}{3}, -\tfrac{14}{3},$$
$$-\tfrac{8}{3}, -\tfrac{11}{3}, +\tfrac{4}{3}, +\tfrac{1}{3}, -\tfrac{5}{3}, -\tfrac{2}{3}, \quad (!) \quad -1, -2, -4. -3, -8 \ldots;$$

the second is finished with $-\frac{1}{3}$; and we prolonged it with 0:

$$\cdots - \tfrac{10}{3}, +\tfrac{16}{3}, +\tfrac{14}{3}, +\tfrac{11}{3}, +\tfrac{5}{3}, +\tfrac{8}{3},$$
$$-\tfrac{7}{3}, -\tfrac{4}{3}, +\tfrac{2}{3}, -\tfrac{1}{3} \quad (!) \quad 0, 1, 3, 2, 7 \ldots.$$

As predecessors to the symbol ! in both pictures we have the former orbits of T, which now become the negative semi-orbits of M, which we glued to integers. Thus the transformation M is now defined on the whole group $\mathbf{Z}_2$. In particular, we have defined M for all integers from $\mathbb{Z}$.

It is clear from the definition that

$$M(-n) = -M(n-1) - 1, \quad n = 1, \ldots.$$

This formula is valid for all $x \in \mathbf{Z}_2$ since $M(-x) = -M(x-1) - 1$, and is true even for the exceptional points $x = -\frac{1}{3}$ and $x = -\frac{2}{3}$ since

$$M(-\tfrac{1}{3}) = -M(-\tfrac{2}{3}) - 1.$$

It is easy to deduce from the definition of M that M is continuous on $\mathbf{Z}_2 \setminus MAX$, and that it is not possible to extend M by continuity to those two points, because, for example, the limit of each of the two sequences $(10)^n(0)^\infty$ and $(10)^n(1)^\infty$ as n tends to infinity, is the same, namely $(10)^\infty(= -\frac{1}{3})$, but the values of M on the sequences tends in the first case to $(1)^\infty(= -\mathbf{1})$ and in the second case to $(0)^\infty(= \mathbf{0})$.

Thus, except for three orbits of T (or four semi-orbits of M), all the other orbits are simultaneously orbits of both transformation. This completely defines the adic realization of the Morse transformation and its orbit partition, as well as the time substitution of the odometer.

6. Conclusion

We gave an explicit form of the time change on the orbits of the odometer in order to obtain the Morse transformation. The answer shows us that the structure of the new ordering of the orbit is *locally finite* in the sense we have defined. We proved that Morse transformation and odometer are allied in the sense of our definition (section 2). This structure of the time change indicates that the two automorphisms are not very different in terms of their orbits (but nevertheless can have different spectrum). It is possible that they have the same *entropy scale* in the sense of [**13**]. What can we say more generally

about the properties of automorphisms which are related by such a time change? For example, if one is Bernoulli will the second also be Bernoulli?

The measure on the space of linear orders of $\mathbb{Z}$ (or on the space of locally finite permutations of $\mathbb{Z}$) which we have defined with our algorithm is of great interest itself. It is the image of the invariant measure on $\mathbf{Z}_2$ with respect to the function $S\{t(k,\cdot)\}_{k\in\mathbb{Z}} : \mathbf{Z}_2 \to \mathfrak{S}_{\mathbb{Z}}$.

It is interesting also to study the time change for other examples of measure-preserving automorphisms, for example the other substitutions like the Morse transformation, and automorphism with positive entropy. The classes of random infinite permutations which appear in these cases give new examples of nontrivial measures on the infinite symmetric group.

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